Arithmetic of quantum entropy function

JHEP08(2009)068

(http://iopscience.iop.org/1126-6708/2009/08/068)

JHEP

Published by IOP Publishing for SISSA

Received: *June 28, 2009*
Accepted: *August 1, 2009*
Published: *August 18, 2009*

Arithmetic of quantum entropy function

Ashoke Sen

Harish-Chandra Research Institute,
Chhatnag Road, Jhusi, Allahabad 211019, India

E-mail: sen@mri.ernet.in

Abstract: Quantum entropy function is a proposal for computing the entropy associated with the horizon of a black hole in the extremal limit, and is related via AdS/CFT correspondence to the dimension of the Hilbert space in a dual quantum mechanics. We show that in $\mathcal{N} = 4$ supersymmetric string theories, quantum entropy function formalism naturally explains the origin of the subtle differences between the microscopic degeneracies of quarter BPS dyons carrying different torsion, i.e. different arithmetical properties. These arise from additional saddle points in the path integral — whose existence depends on the arithmetical properties of the black hole charges — constructed as freely acting orbifolds of the original $AdS_2 \times S^2$ near horizon geometry. During this analysis we demonstrate that the quantum entropy function is insensitive to the details of the infrared cutoff used in the computation, and the details of the boundary terms added to the action. We also discuss the role of the asymptotic symmetries of AdS_2 in carrying out the path integral in the definition of quantum entropy function. Finally we show that even though quantum entropy function is expected to compute the absolute degeneracy in a given charge and angular momentum sector, it can also be used to compute the index. This can then be compared with the microscopic computation of the index.

Keywords: Black Holes in String Theory, D-branes, AdS-CFT Correspondence, String Duality

ArXiv ePrint: 0903.1477

doi:10.1088/1126-6708/2009/08/068

JHEP08(2009)068

Contents

JHEP08(2009)068

1 Introduction

Extremal black holes[1] provide us with a very useful laboratory for understanding the quantum aspects of black hole physics in string theory [3]. In particular one expects that for supersymmetric extremal black holes one should be able to make a precise comparison between the macroscopic and the microscopic entropies. Let $d_{\rm micro}(\vec{q})$ denote the degeneracy of BPS microstates carrying total charge $\vec{q}$ in any string theory. Then on general grounds one expects the following relation between $d_{\rm micro}(\vec{q})$ and the macroscopic quantities associated with the black hole:

$$d_{\rm micro}(\vec{q}) = d_{\rm macro}(\vec{q})\,, \tag{1.1}$$

$$d_{\rm macro}(\vec{q}) = \sum_n \sum_{\substack{\{\vec{q}_i\},\vec{q}_{\rm hair} \\ \sum_{i=1}^n \vec{q}_i + \vec{q}_{\rm hair} = \vec{q}}} \left\{ \prod_{i=1}^n d_{\rm hor}(\vec{q}_i) \right\} d_{\rm hair}(\vec{q}_{\rm hair}; \{\vec{q}_i\})\,. \tag{1.2}$$

The n-th term on the right hand side of (1.2) represents the contribution to the degeneracy from an n-centered black hole configuration. $d_{\rm hor}(\vec{q}_i)$ is the degeneracy associated with the horizon of a single centered black hole (or any other black object) carrying charge $\vec{q}_i$, and $d_{\rm hair}(\vec{q}_{\rm hair}; \{\vec{q}_i\})$ is the degeneracy of the hair [4], carrying total charge $\vec{q}_{\rm hair}$, of an n-centered black hole whose horizons carry charges $\vec{q}_1, \vec{q}_2, \ldots, \vec{q}_n$. In order to make (1.2) concrete we shall always work in a fixed duality frame. Once such a frame is fixed, the notion of a classical solution has a well defined meaning: it is a solution to the classical equations of motion without any external source term. A black hole in this duality frame will refer to

[1] Throughout this paper we shall use the word extremal black hole to denote the extremal limit of a non-extremal black hole as reviewed in [1, 2].

a solution to the *classical equations of motion* with non-singular near horizon geometry. On other other hand the 'hair' of a black hole will refer to normalizable fluctuations of the black hole solution with support outside the black hole horizon. For BPS black holes $d_{\rm hair}$ can be computed by quantizing these normalizable fluctuations and identifying the subset of states which satisfy the BPS condition. Since the space of normalizable fluctuations of a (multi-centered) black hole could depend on the charges carried by various centers, $d_{\rm hair}$ can depend on the charge $\vec{q}_{\rm hair}$ of the hair as well as the charges $\{\vec{q}_i\}$ of the horizons. On the other hand since an infinite throat separates each horizon from the rest of the space-time, we expect the degeneracy associated with the i-th horizon to depend only on the charge $\vec{q}_i$ carried by that horizon, and not on the charges carried by the other horizons or the hair.

Our main focus in this paper will be on $d_{\rm hor}(\vec{q})$. Quantum entropy function (QEF) is a proposal for computing $d_{\rm hor}(\vec{q})$ in terms of a path integral over string fields on the near horizon attractor geometry of the black hole containing a product of AdS_2 and a compact space K. This has been described in (2.6). We shall use (2.6) to analyze the QEF of quarter BPS black holes in heterotic string theory compactified on T^6 and compare this with the microscopic prediction. This theory has $O(6,22;\mathbb{Z})$ T-duality group which is generated by $O(6,22;\mathbb{R})$ matrices preserving the 28 dimensional Narain lattice [5], and the dyons are characterized by 28 dimensional electric and magnetic charge vectors (Q,P) taking values in the Narain lattice. It has been known for sometime that the microscopic degeneracy of the dyons, besides depending on the invariants $(Q^2,P^2,Q\cdot P)$ of the continuous $\mathrm{SO}(6,22;\mathbb{R})$ group, also depends on

$$\ell = \gcd\{Q_iP_j - Q_jP_i\}\,, \tag{1.3}$$

which is an invariant of the discrete T- and S-duality groups [6–8] encoding arithmetical information about the charges. We shall refer to this integer as torsion. Using an appropriate S-duality transformation a dyon of torsion ℓ can be brought to the form $(\ell Q_0, P_0)$ where Q_0, P_0 are primitive lattice vectors satisfying $\gcd\{Q_{0i}P_{0j} - Q_{0j}P_{0i}\} = 1$ [8]. The degeneracy of such dyons is given by [9–11]

$$(-1)^{Q\cdot P+1}\sum_{s|\ell} s\, f(Q^2/s^2, P^2, Q\cdot P/s)\,, \tag{1.4}$$

where $(-1)^{Q\cdot P+1} f(Q^2,P^2,Q\cdot P)$ denotes the degeneracy of a dyon of charge (Q,P) with $\gcd\{Q_iP_j - Q_jP_i\} = 1$. Our goal will be to understand this formula from a macroscopic viewpoint, i.e. by using the quantum entropy function.[2]

The function f is given by the triple Fourier transform of the inverse of the Igusa cusp form [15–19]. In principle QEF should provide a complete macroscopic derivation of the function f as well as the structure of eq. (1.4). However our goal in this paper will be modest; instead of providing a detailed derivation of the function f from QEF, we shall simply try to identify the origin of different terms in the sum in (1.4). We shall show that there are natural candidates which reproduce these terms, — they reflect contribution from different saddle points with the same asymptotic field configuration as the near horizon attractor geometry of the black hole. These new saddle points, obtained

[2]For earlier discussion on the relation between attractor geometry and arithmetic see [12–14].

JHEP08(2009)068

as freely acting $\mathbb{Z}_s$ orbifolds of the original near horizon attractor geometry, are in one to one correspondence with the divisors of ℓ, and furthermore the classical contributions to QEF from these saddle points coincide with the leading asymptotic behaviour of the summands $f(Q^2/s^2, P^2, Q\cdot P/s)$ for large charges.

In this context we would like to remind the reader that the function $f(Q^2, P^2, Q\cdot P)$ itself can be expressed as a sum over infinite number of terms, associated with the infinite number of poles of the Igusa cusp form, and it was argued in [20] that these different terms can be associated with different saddle points obtained by taking the quotient of the original $AdS_2\times S^2$ background by $\mathbb{Z}_N$ orbifold groups. These orbifolds exist for all charges, including those with unit torsion. The saddle points considered here are distinct from the ones used in [20] due to inclusion of additional shift transformations in the orbifold group action, and exist only for dyons carrying non-trivial torsion.

The analysis of QEF for dyons of $\mathcal{N}=4$ supersymmetric string theory has been described in section 6. However we also address several technical issues related to the computation of QEF, a summary of which is given below.

1. Since AdS_2 has infinite volume, path integral over string fields in AdS_2 suffers from infrared divergences as in the case of higher dimensional AdS spaces [21, 22]. Thus in order to get sensible answers one first needs to use an infrared cutoff that regularizes the volume of AdS_2 and at the end of the computation take the cutoff to infinity. In earlier work [1, 2] the infrared cutoff was chosen in a way so that it preserves an SO(2) subgroup of the SL$(2,R)$ isometry of AdS_2. However one can choose a more general infrared cutoff that destroys all isometries of AdS_2. We show that even with such general infrared cutoff one gets exactly the same value of QEF. Thus QEF is insensitive to the details of the infrared cutoff. We also show that QEF is insensitive to the details of the boundary terms which are added to the action.

2. Besides the SL$(2,R)$ isometry, AdS_2 has an infinite group of asymptotic symmetries. We need to take special care in defining the path integral so that integration over these symmetry directions do not generate an infinite factor. We discuss this in detail and give a specific prescription for removing these infinite factors from the path integral.

3. Via AdS/CFT correspondence [21–24] one can argue that QEF counts the number of ground states of the black hole in a given charge and angular momentum sector after removing the contribution from the hair degrees of freedom [2]. Now often in the comparison between the macroscopic and the microscopic entropies one computes an index rather than the absolute degeneracy on the microscopic side since it is this index that is protected by supersymmetry. In particular the 'degeneracy formula' given in (1.4) actually refers to the sixth helicity trace $-B_6$ [25, 26]. This raises a question as to how QEF can be compared with the index computed on the microscopic side. We show that since QEF measures the degeneracy in fixed charge and angular momentum sector, it can actually be used to compute an index on the macroscopic side. This can then be compared with the microscopic index.

JHEP08(2009)068

2 Quantum entropy function

In this section we shall give a brief overview of the quantum entropy function, — the quantity that is supposed to compute $d_{\rm hor}(\vec{q})$ appearing in (1.2). We begin by writing down the background fields describing the AdS_2 near horizon geometry of an extremal black hole [1, 2]:

$$ds^2 = v\left(-(r^2-1)dt^2 + \frac{dr^2}{r^2-1}\right), \quad F^{(i)}_{\rm rt} = e_i, \quad \cdots \tag{2.1}$$

where $F^{(i)}_{\mu\nu} = \partial_\mu A^{(i)}_\nu - \partial_\nu A^{(i)}_\mu$ are the gauge field strengths, v and e_i are constants and $\cdots$ denotes near horizon values of other fields. Under euclidean continuation

$$t = -i\theta\,, \tag{2.2}$$

we have

$$ds^2 = v\left((r^2-1)d\theta^2 + \frac{dr^2}{r^2-1}\right), \quad F^{(i)}_{r\theta} = -i\,e_i, \quad \cdots \tag{2.3}$$

Under a further coordinate change

$$r = \cosh\eta\ , \tag{2.4}$$

(2.3) takes the form

$$ds^2 = v\ \left(d\eta^2 + \sinh^2\eta\, d\theta^2\right), \qquad F^{(i)}_{\eta\theta} = -ie_i\ \sinh\eta, \qquad \cdots .$$

The metric is non-singular at the point $\eta = 0$ if we choose θ to have period 2π. Integrating the field strength we can get the form of the gauge field:

$$A^{(i)}_\mu dx^\mu = -i\,e_i\,(\cosh\eta - 1)\,d\theta = -i\,e_i\,(r-1)\,d\theta\,. \tag{2.5}$$

Note that the -1 factor inside the parenthesis is required to make the gauge fields non-singular at $\eta = 0$. In writing (2.5) we have chosen $A_\eta = 0$ gauge.

Quantum entropy function is now defined as

$$d_{\rm hor}(\vec{q}) = \left\langle \exp[-iq_i \oint d\theta\, A^{(i)}_\theta] \right\rangle^{\rm finite}_{AdS_2}\ . \tag{2.6}$$

Here $\oint d\theta\, A^{(i)}_\theta$ denotes the integral of the i-th gauge field along the boundary of AdS_2. $\langle\ \rangle_{AdS_2}$ denotes the unnormalized path integral over various fields, satisfying the same asymptotic behaviour as (2.5), weighted by e^{-A} where A is the Euclidean action. The path integral must be performed over all the fields in string theory. The superscript '$finite$' refers to the finite part of the amplitude defined as follows. If we regularize the infrared divergence by putting an explicit upper cutoff on r (or η), and denote by L the length of the boundary of this regulated AdS_2, then for large cutoff, i.e. large L, the amplitude has the form $e^{CL+\mathcal{O}(L^{-1})} \times \Delta$ where C and Δ are L-independent constants. The finite part of the amplitude is defined to be the constant Δ. Eq. (2.6), together with (1.1), (1.2), gives a precise relation between the microscopic degeneracy and an appropriate partition function in the near horizon geometry of the black hole. We call the right hand side of (2.6), i.e. the constant Δ, the 'quantum entropy function' (QEF).

JHEP08(2009)068

In defining the path integral over AdS_2 we need to put boundary conditions on various fields. Special care is needed to fix the boundary condition on $A_\theta^{(i)}$. In the $A_\eta^{(i)} = 0$ gauge the solution of the linearized Maxwell's equations around this background has two independent solutions near the boundary: $A_\theta^{(i)} =$ constant and $A_\theta^{(i)} \propto r$. Since the latter is the dominant mode for large r, we put boundary condition on the latter mode, allowing the constant mode of the gauge field to be integrated over. This is done formally by requiring

$$\lim_{r\to\infty} \frac{\delta A_{\rm bulk}}{\delta F_{r\theta}^{(i)}} = -i\, q_i \tag{2.7}$$

where $\{q_i\}$ are fixed numbers and $A_{\rm bulk}$ is the full bulk action of the two dimensional theory. This corresponds to fixing the charges carried by the black hole to be $\{q_i\}$. In this case under an infinitesimal variation of the gauge field component $A_\theta^{(i)}$ we have

$$-\delta A_{\rm bulk} = -\int dr\, d\theta\, \partial_r \delta A_\theta^{(i)} \frac{\delta A_{\rm bulk}}{\delta F_{r\theta}^{(i)}} = \int dr\, d\theta\, \delta A_\theta^{(i)} \partial_r \frac{\delta A_{\rm bulk}}{\delta F_{r\theta}^{(i)}} - \oint d\theta\, \delta A_\theta^{(i)} \frac{\delta A_{\rm bulk}}{\delta F_{r\theta}^{(i)}}\Big|_{\rm boundary} . \tag{2.8}$$

Setting the first term on the right hand side of (2.8) to zero yields the equation of motion of $A_\theta^{(i)}$. On the other hand the second term cancels against the variation of the $-iq_i \oint d\theta\, A_\theta^{(i)}$ term in the exponent in (2.6). The boundary conditions on the other fields are fixed in the standard manner, e.g. in the $g_{\eta\eta} = v$, $g_{\theta\eta} = 0$ gauge we freeze the mode of $g_{\theta\theta}$ proportional to r^2 to its value given in (2.5) and allow the constant mode of the metric to fluctuate. On the other hand for massless scalar fields we require the constant mode of the fields to vanish at asymptotic infinity. Appropriate boundary terms must be added to the action so that the variation of the action under an arbitrary variation of various fields, subject to these boundary conditions, vanishes when equations of motion are satisfied.

In this context note that we could have included the exponential term in (2.6) as part of the boundary term in the action and expressed (2.6) as

$$d_{\rm hor}(\vec{q}) = Z_{AdS_2}^{\rm finite} , \tag{2.9}$$

where Z_{AdS_2} is the partition function of AdS_2 computed using the natural boundary condition that fixes the electric charge rather than the gauge potential. We shall continue to use the notation (2.6) since it explicily displays the part of the boundary term that requires the gauge potential and cannot be expressed in terms of gauge field strengths. (2.6) also has the advantage that it explicitly displays the dependence on the charges q_i.

Since QEF can be regarded as the partition function on an Euclidean AdS_2 in a fixed charge (including angular momentum) sector, the usual rules of AdS/CFT correspondence [21, 22] tells us that it measures the partition function of a dual quantum mechanics living on the boundary of AdS_2 [2]. This dual quantum mechanics in turn can be obtained as the infrared limit of the quantum mechanics describing the dynamics of the black hole after removing its hair degrees of freedom. Since from microscopic analysis one finds that the black hole has a gap separating the ground state from the first excited state, only the ground states in a given charge and angular momentum sector survive in the infrared limit, and the partition function takes the form $d(q)e^{-E_0 L}$ where $d(q)$ is the ground state

JHEP08(2009)068

degeneracy, E_0 is the ground state energy and L is the length of the boundary of AdS_2. The 'finite part' of this is given by $d(q)$. Thus QEF should count the number of ground states of the black hole in a given charge and angular momentum sector after removing the contribution from the hair degrees of freedom. From this viewpoint the proposal that QEF measures the degeneracy associated with the horizon is a direct consequence of AdS_2/CFT_1 correspondence. As a consistency check on this proposal, it was shown in [2] that QEF reduces to the exponential of the Wald entropy [27–30] in the classical limit.

3 Insensitivity to the infrared cutoff

It was shown in [2] that QEF reduces to the exponential of the Wald entropy in the classical limit. During this proof the infrared divergence associated with the infinite volume of AdS_2 was regularized by putting a cutoff at $r = r_0$. In this section we shall consider a more general cutoff of the form[3]

$$r \le r_0\, f(\theta)\,, \quad f(\theta+2\pi) = f(\theta), \quad f(\theta) > 0, \quad \int_0^{2\pi} d\theta f(\theta) = 2\pi\,, \tag{3.1}$$

for any smooth function $f(\theta)$ and show that the result does not change. We begin by noting that in the classical limit QEF is given by the finite part of

$$\exp\left(-A_{\rm bulk} - A_{\rm boundary} - iq_i \oint A^{(i)}_\theta\, d\theta\right)\,, \tag{3.2}$$

where $A_{\rm bulk}$ and $A_{\rm boundary}$ represent contributions from the bulk and the boundary terms in the classical action in the background (2.5). If $\mathcal{L}$ denotes the Lagrangian density of the two dimensional theory, then the bulk contribution to the action in the background (2.5) takes the form:

$$\begin{aligned} A_{\rm bulk} &= -\int d^2x\, \sqrt{\det g}\, \mathcal{L} \\ &= -\int_0^{2\pi} d\theta \int_0^{\cosh^{-1}(r_0 f(\theta))} d\eta\, \sinh\eta\, v\, \mathcal{L} \\ &= -v\,\mathcal{L}\int_0^{2\pi} (r_0\, f(\theta) - 1)\, d\theta \\ &= -2\pi\, v\, \mathcal{L}\, (r_0 - 1)\,. \end{aligned} \tag{3.3}$$

In going from the second to the third step in (3.3) we have used the fact that due to the SO(2, 1) invariance of the AdS_2 background, $\mathcal{L}$ is independent of η and θ. In this parametrization the length L of the boundary is given by

$$L = \int_0^{2\pi} d\theta\, \sqrt{r_0^2 f(\theta)^2 - 1} = 2\pi\, r_0 + \mathcal{O}(r_0^{-1})\,. \tag{3.4}$$

[3] We have imposed the last condition in (3.1) to fix the overall normalization of f, since any change in the normalization of f can be absorbed into a redefinition of r_0.

JHEP08(2009)068

The contribution from the last term in the exponent in (3.2) can also be calculated easily using the expression for $A^{(i)}_\theta$ given in (2.5). We get

$$iq_i \oint A^{(i)}_\theta \, d\theta = 2\pi \, \vec{q}\cdot\vec{e}\,(r_0 - 1)\,. \tag{3.5}$$

Finally, the contribution from $A_{\rm boundary}$ can be analyzed as follows. $A_{\rm boundary}$ is chosen so that under an arbitrary variation of the fields the boundary terms arising in $\delta A_{\rm bulk}$ get cancelled by the boundary terms arising from the variation of $A_{\rm boundary}$. Now as we have discussed in section 2, the boundary terms in $\delta A_{\rm bulk}$ proportional to $\delta A^{(i)}_\theta$ are cancelled by the last term in the exponent in (3.2). Thus $A_{\rm boundary}$ must be chosen so as to cancel the boundary contribution to $\delta A_{\rm bulk}$ from the variation of the other fields, without giving any further term involving $\delta A^{(i)}_\theta$. This in particular requires that the dependence of $A_{\rm boundary}$ on $A^{(i)}_\theta$ enters only through the field strengths $F^{(i)}_{\theta\eta}$. In order to analyze the contribution from these terms we choose new coordinates near the boundary[4]

$$\xi = \eta - \ln r_0 - \ln 2 - \ln f(\theta), \qquad w = r_0 \int_0^\theta f(u)du - \frac{1}{2} r_0^{-1}\, e^{-2\xi}\, \frac{f'(\theta)}{f(\theta)^2}\,, \tag{3.6}$$

so that we have, due to the equivalence $\theta \equiv \theta + 2\pi$, and the properties of $f(\theta)$ described in (3.1),

$$w \equiv w + 2\pi r_0\,, \tag{3.7}$$

and the boundary $\eta = \cosh^{-1}(r_0 f(\theta))$ is at

$$\xi = \mathcal{O}(r_0^{-2})\,. \tag{3.8}$$

In this coordinate system the background (2.5) near the boundary takes the form

$$\begin{aligned} ds^2 &= v\,\left(d\xi^2 + e^{2\xi}\, dw^2\right) + \mathcal{O}(r_0^{-2}), \\ F^{(i)}_{\xi w} &= -ie_i\, e^{\xi} + \mathcal{O}(r_0^{-2})\,, \quad \cdots\,. \end{aligned} \tag{3.9}$$

We now note that the background (3.9) is independent of r_0 up to corrections of order r_0^{-2}. Thus the integrand of the boundary term is r_0 independent up to terms of order r_0^{-2}. (3.9) also has the translation symmetry under $w \to w + c$ up to corrections of order r_0^{-2} so that the w integral in $A_{\rm boundary}$ will produce a factor of $L = 2\pi r_0$ up to corrections of order r_0^{-1}. Thus the boundary term will give a contribution of the form

$$A_{\rm boundary} = 2\pi r_0\, K + \mathcal{O}(r_0^{-1})\,, \tag{3.10}$$

for some constant K. Combining (3.3), (3.5) and (3.10) we get

$$\begin{aligned} \exp\left(-A_{\rm bulk} - A_{\rm boundary} - iq_i \oint A^{(i)}_\theta \, d\theta\right) &= \exp\left[-2\pi r_0(\vec{q}\cdot\vec{e} - v\,\mathcal{L} + K) + \mathcal{O}(r_0^{-1})\right] \\ &\quad \times \exp\left[2\pi(\vec{q}\cdot\vec{e} - v\,\mathcal{L})\right]\,. \end{aligned} \tag{3.11}$$

[4] Although it is not necessary, we could choose the coordinate transformations near $\eta = 0$ to look like $\xi = \eta - \ln r_0 - \ln 2$, $w = 2\pi r_0\theta$, so that the solution near the core is independent of the choice of $f(\theta)$.

JHEP08(2009)068

Thus QEF, given by the finite part of (3.11), takes the form

$$d_{\rm hor}(q) = \exp\left[2\pi(\vec{q}\cdot\vec{e} - v\,\mathcal{L})\right] . \tag{3.12}$$

The right hand side of (3.12) is the exponential of the Wald entropy [31]. Thus we see that the result is independent of the function $f(\theta)$ we use to regulate the infrared divergence. Also note that by changing the boundary action we can change the coefficient K, and hence the coefficient of the term proportional to r_0 in the exponent of (3.11), but the r_0 independent part that contributes to QEF is not affected by the choice of the boundary action.[5]

Next we turn to the proof of $f(\theta)$ independence of the QEF after inclusion of quantum corrections. For this we need to show that the action evaluated for an off-shell field configuration is also independent of the choice of $f(\theta)$. Now the allowed off-shell field configurations over which we carry out the path integral are constrained by the boundary conditions on the various fields. We choose the boundary conditions so as to allow only normalizable deformations of the original background. In particular the difference between the action of the deformed field configuration and the original action must be finite. This requires that the additional contribution $\Delta\left(\sqrt{\det g}\,\mathcal{L}\right)$ in the lagrangian density, that arises due to the difference between the general off-shell field configuration and the original $AdS_2\times K$ background, must fall off faster than r^{-1} for large r. To see if such a $\Delta\left(\sqrt{\det g}\,\mathcal{L}\right)$ can give $f(\theta)$ dependent contribution to the action we can compute the difference between the contributions for a general $f(\theta)$ and $f(\theta)=1$. This is given by

$$\int_0^{2\pi} d\theta \int_{r_0}^{r_0 f(\theta)} dr\, \Delta\left(\sqrt{\det g}\,\mathcal{L}\right) . \tag{3.13}$$

Since for large r, $\Delta\left(\sqrt{\det g}\,\mathcal{L}\right)$ falls of faster than r^{-1}, the integrand falls off faster than r_0^{-1} in the domain of integration. On the other hand the size of the domain is of order r_0. Thus (3.13) vanishes in the $r_0\to\infty$ limit. This shows that even the off-shell action is independent of the choice of $f(\theta)$.

Since the above argument has been somewhat abstract, we shall now illustrate explicitly how it works for the terms involving the metric, gauge fields and scalars. We choose the following boundary conditions on the metric and the gauge fields (see e.g. [33])

$$\begin{aligned}
&g_{\eta\eta} = v + \mathcal{O}(e^{-2\eta}), \qquad g_{\eta\theta} = \mathcal{O}(e^{-2\eta}), \qquad g_{\theta\theta} = \frac{v}{4}\,e^{2\eta} + \mathcal{O}(1),\\
&A^{(i)}_\theta = -\frac{1}{2}\,i\,e_i\,e^{\eta} + \mathcal{O}(1), \quad A^{(i)}_\eta = \mathcal{O}(e^{-2\eta}), \qquad F^{(i)}_{\eta\theta} = -\frac{1}{2}\,i\,e_i\,e^{\eta} + \mathcal{O}(e^{-\eta}) .
\end{aligned} \tag{3.14}$$

This background differs from the AdS_2 background by terms which fall off as powers of $e^{-\eta}$ for large η. We shall refer to these terms as subleading terms. We shall now show that in the $r_0\to\infty$ limit the contribution from the subleading terms in (3.14) to the action

[5] We could also try to adjust the boundary terms, i.e. the constant K, so that the term in the exponent of (3.11) linear in r_0 vanishes, and we are left with only the finite part of the amplitude in the $r_0\to\infty$ limit. The situation is different in the case of higher dimensional AdS space-times, e.g. for AdS_3, where the boundary terms cannot be adjusted to cancel all the divergences coming from the bulk terms, reflecting the effect of the central charge [32].

JHEP08(2009)068

is independent of $f(\theta)$. On the other hand the leading contribution to the action, being the same as that from an AdS_2 background, has already been shown to be independent of the choice of $f(\theta)$. This would then establish that the off-shell action computed for an arbitrary configuration satisfying (3.14) is independent of $f(\theta)$ and hence the full QEF is also independent of $f(\theta)$.

First we analyze the subleading contribution to $A_{\rm bulk}$. For this we express the difference between the bulk action for a general $f(\theta)$ and $f(\theta) = 1$ as

$$\Delta A_{\rm bulk} = -\int_0^{2\pi} d\theta \int_{r_0}^{r_0 f(\theta)} dr \, \sqrt{\det g} \;\; \mathcal{L} \, . \tag{3.15}$$

Now the integration volume is of order r_0 and the leading order contribution to $\sqrt{\det g}\,\mathcal{L}$ is of order unity. On the other hand the subleading terms in every field given in (3.14) is suppressed by a factor of $e^{-2\eta}$ compared to the leading term. The only exception is the constant contribution to $A^{(i)}_\theta$, but this does not contribute to the field strength, and $\mathcal{L}$ depends on the $A^{(i)}_\mu$ only through its field strength. Thus the net contribution to $\mathcal{L}$ from the subleading terms in (3.14) is of order $e^{-2\eta}$ for large η. This is of order r_0^{-2} in the domain of integration given in (3.15), and gives a contribution of order r_0^{-1} to $\Delta A_{\rm bulk}$. This vanishes in the $r_0 \to \infty$ limit. There is of course a finite contribution to the $A_{\rm bulk}$ from the subleading terms from the region where r (i.e. η) is finite, but this does not depend on the choice of $f(\theta)$.

An identical argument shows that the contribution from the subleading terms to $A_{\rm boundary}$ is suppressed by powers of r_0^{-1} both at $r = r_0$ and at $r = r_0\, f(\theta)$. Finally the term proportional to $\int_0^{2\pi} A^{(i)}_\theta \, d\theta$ may be analyzed as follows. We note that

$$\int_0^{2\pi} A^{(i)}_\theta d\theta\Big|_{r=r_0 f(\theta)} - \int_0^{2\pi} A^{(i)}_\theta d\theta\Big|_{r=r_0} = \int_0^{2\pi} d\theta \int_{r_0}^{r_0 f(\theta)} dr F^{(i)}_{r\theta} = \int_0^{2\pi} d\theta \int_{\cosh^{-1} r_0}^{\cosh^{-1}(r_0 f(\theta))} d\eta F^{(i)}_{\eta\theta} \, . \tag{3.16}$$

Now since the subleading contribution to $F^{(i)}_{\eta\theta}$ is of order $e^{-\eta} \sim r_0^{-1}$ and the η integration spans a range of order unity, we conclude that the contribution to the right hand side of (3.16) from the subleading terms in (3.14) is of order r_0^{-1} and hence vanishes in the $r_0 \to \infty$ limit. This shows that in the $r_0 \to \infty$ limit the contribution to the full action from the subleading terms is independent of $f(\theta)$.

So far in our discussion we have ignored the scalar fields. If there are massless scalar fields, then the natural boundary conditions on these fields may be found by examining the solutions to the linearized equations of motion near the boundary and then allowing the normalizable mode to fluctuate. This gives, for a minimally coupled scalar field,

$$\phi_s = u_s + \mathcal{O}(e^{-\eta}) \, , \tag{3.17}$$

where u_s is the attractor value. Thus the subleading corrections to ϕ_s are suppressed by powers of $e^{-\eta}$ instead of $e^{-2\eta}$, and could invalidate our earlier argument. However since the attractor geometry extremizes the action with respect to ϕ_s, we expect that the correction to the action from the subleading terms is at least quadratic in $(\phi_s - u_s)$ and

JHEP08(2009)068

other fluctuations and hence is again suppressed by powers of $e^{-2\eta}$. Similarly the boundary terms in the action must also be quadratic in the fluctuations and are suppressed by powers of $e^{-2\eta}$. This in turn shows that the off-shell action is independent of the choice of $f(\theta)$ even after the inclusion of the subleading terms. Hence QEF is independent of the choice of $f(\theta)$ even after inclusion of quantum corrections.

Finally we would like to note that our discussion has been centered around studying the effect of integration over the massless fields. In principle the contribution from integration over the massive fields can be analyzed in the same way. Alternatively we can integrate out the massive fields from the beginning and work with an effective Lagrangian density $\mathcal{L}_{\rm eff}$ involving the massless fields. Whatever we have done could then be repeated by replacing $\mathcal{L}$ by $\mathcal{L}_{\rm eff}$, and the final result — that QEF is insensitive to the details of the infrared cut-off — would continue to hold.

4 Effect of asymptotic symmetries

In this section we shall analyze the asymptotic symmetries of string theory in the near horizon background of an extremal black hole and their role in defining the path integral over the string fields in the AdS_2 background [34, 35]. For this we consider the class of field configurations satisfying (3.14), and identify diffeomorphisms accompanied by gauge transformations for which the asymptotic conditions given in (3.14) are preserved. The following diffeomorphism plus gauge transformation satisfy this restriction

$$\begin{aligned}
\theta &= \chi(\widetilde\theta) - 2\, e^{-2\widetilde\eta}\chi''(\widetilde\theta) + \mathcal{O}(e^{-4\widetilde\eta}), \qquad \eta = \widetilde\eta - \ln\chi'(\widetilde\theta) + \mathcal{O}(e^{-2\widetilde\eta}), \\
\Lambda^{(i)} &= -2\, i\, e_i\, e^{-\widetilde\eta}\frac{\chi''(\widetilde\theta)}{\chi'(\widetilde\theta)} + \Lambda_0^{(i)}(\widetilde\theta) + \mathcal{O}(e^{-2\widetilde\eta})\,,
\end{aligned} \tag{4.1}$$

where $\chi(\widetilde\theta)$ and $\Lambda_0^{(i)}(\widetilde\theta)$ are some functions satisfying

$$\chi(\widetilde\theta + 2\pi) = \chi(\widetilde\theta) + 2\,\pi\,, \qquad \Lambda_0^{(i)}(\widetilde\theta + 2\pi) = \Lambda_0^{(i)}(\widetilde\theta)\,, \qquad \chi'(\widetilde\theta) > 0\,. \tag{4.2}$$

$\chi(\widetilde\theta)$ and $\Lambda_0^{(i)}(\widetilde\theta)$ generate global diffeomorphism and global gauge transformations respectively. The subleading terms in (4.1) can be used to locally fix the gauge

$$g_{\widetilde\eta\widetilde\eta} = 1, \quad g_{\widetilde\eta\widetilde\theta} = A^{(i)}_{\widetilde\eta} = 0\,, \tag{4.3}$$

but we shall proceed without making any specific choice of gauge. If we denote by $\widetilde\chi(\theta)$ the inverse transformation of g, i.e. $\widetilde\theta = \widetilde\chi(\theta)$, then we have

$$\widetilde\chi(\theta + 2\,\pi) = \widetilde\chi(\theta) + 2\,\pi, \qquad \widetilde\chi'(\theta) = 1/\chi'(\widetilde\theta) > 0\,, \tag{4.4}$$

and the transformations (4.1) may be inverted as

$$\begin{aligned}
\widetilde\theta &= \widetilde\chi(\theta) - 2\, e^{-2\eta}\widetilde\chi''(\theta) + \mathcal{O}\left(e^{-4\eta}\right)\,, \qquad \widetilde\eta = \eta - \ln\widetilde\chi'(\theta) + \mathcal{O}(e^{-2\eta}), \\
\widetilde\Lambda^{(i)} &= -\Lambda^{(i)} = -2\, i\, e_i\, e^{-\eta}\frac{\widetilde\chi''(\theta)}{\widetilde\chi'(\theta)} + \widetilde\Lambda_0^{(i)}(\theta) + \mathcal{O}(e^{-2\eta})\,,
\end{aligned} \tag{4.5}$$

where $\widetilde\Lambda_0^{(i)}(\theta) = -\Lambda_0^{(i)}(\widetilde\theta)$.

JHEP08(2009)068

Let us now view (4.1) as an active transformation and consider two field configurations related to each other by such a transformation. Naively the action, being diffeomorphism invariant, will have the same value for these two configurations. However we should remember that the action is divergent and must be regulated by putting a cutoff. Thus for a fixed cutoff the new configuration generated by the transformation (4.1) has *a priori* a different action than the one for the original background given in (2.5), (2.5). We shall denote by $S[\chi]$ the action associated with the new configuration with a cutoff at $\widetilde{\eta} = \eta_0$, — then the action associated with the original background can be obtained by setting $\chi(\widetilde{\theta}) = \widetilde{\theta}$ in $S[\chi]$. We can compute $S[\chi]$ by using the transformation (4.5) to map the configuration back to the original configuration. This will change the cutoff to

$$\eta = \eta_0 + \ln \widetilde{\chi}'(\theta) + \mathcal{O}(e^{-2\eta_0}) \,, \tag{4.6}$$

or equivalently, in the $r = \cosh\eta$ coordinate

$$r = r_0\, \widetilde{\chi}'(\theta) + \mathcal{O}(r_0^{-1}), \qquad r_0 \equiv \cosh\eta_0 \,. \tag{4.7}$$

Since the configuration expressed in the (η, θ) coordinate system is χ-independent, the only possible dependence of the action on χ comes through the χ dependence of the cutoff given in (4.7). This is precisely the problem addressed in section 3, with $f(\theta)$ replaced by $\widetilde{\chi}'(\theta)$. In particular due to eq. (4.4) we have $\int d\theta \widetilde{\chi}'(\theta) = 2\pi$, so that the last condition in (3.1) is satisfied. We can now use the result of section 3 to infer that $S[\chi]$ is independent of χ. This allows us to declare diffeomorphisms of the type described in (4.1) as gauge transformations and restrict the path integral to over configurations which are not related to each other by diffeomorphisms of the type given in (4.1). In fact we are forced to do so, since otherwise summing over configurations related to each other by transformations (4.1) will produce an infinite factor in the path integral. Similar remarks hold for gauge transformations generated by $\Lambda^{(i)}(\theta)$.

We cannot however declare all transformations of the type (4.1) as gauge transformations. If we do so then for AdS_2 background, which is invariant under an $\mathrm{SL}(2,R)$ subgroup of the transformations (4.1) generated by

$$\delta\, w = \frac{i}{2}(1+w)^2\epsilon_{-1} - \frac{1}{2}(1-w^2)\epsilon_0 - \frac{i}{2}(1-w)^2\epsilon_1, \qquad w \equiv \sqrt{\frac{r-1}{r+1}}\, e^{i\theta} \,, \tag{4.8}$$

we need to divide the path integral by the volume of the $\mathrm{SL}(2,R)$ group. Due to the infinite volume of the $\mathrm{SL}(2,R)$ group the result will vanish. To remedy this we declare transformations of the type (4.1), modulo an $\mathrm{SL}(2,R)$ transformation, as pure gauge transformations.[6] In that case we do not need to divide the path integral over AdS_2 background by the volume of $\mathrm{SL}(2,R)$ group, and this gives a finite result.

In supersymmetric theories there can be fermionic symmetries which leave the action invariant, and we should retrict the path integral so as not to integrate over these symmetry directions; otherwise the path integral will vanish due to integration over these fermionic

[6] In practice this can be achieved e.g. by requiring the transformations (4.1) to leave fixed three points on the boundary of AdS_2.

JHEP08(2009)068

zero modes. However in this case one could also try to use an alternate approach in which the infinities arising from integration over the bosonic zero modes may be canceled against the zeroes coming from the integration over the fermionic zero modes. In this case we do not need to declare the transformations (4.1) and their fermionic cousins as pure gauge transformations.

5 Index or degeneracy?

One of the issues which arise in comparing the microscopic and the macroscopic entropies is that in the microscopic theory we typically compute the helicity trace index while the Bekenstein-Hawking entropy or Wald entropy is supposed to compute the logarithm of the absolute degeneracy. So how can we compare the two quantities? We shall now argue that quantum entropy function formalism provides a natural resolution of this puzzle. For definiteness we shall focus on four dimensional black holes, but similar analysis can be carried out in other dimensions. In four dimensions the relevant index is the helicity trace index [25, 26] $B_{2k} = (-1)^k\, Tr\left[(-1)^{2J}(2J)^{2k}\right]/(2k)!$ where J denotes the helicity of the state (or component of angular momentum along some specific direction in the rest frame) and $4k$ is equal to the number of supersymmetry generators which are broken by the black hole. For quarter BPS dyons in $\mathcal{N} = 4$ supersymmetric string theories in four dimensions, $4k = 12$.

Now suppose that in (1.1) we replace the left hand side by such a helicity trace index. Then on the right hand side also we should compute the helicity trace index. Now the total angular momentum J carried by the black hole is a sum of the angular momentum from the horizon and the hair. Thus the $(-1)^{2J}(2J)^{2k}$ factor will be replaced by $(-1)^{2J_{\rm hor}+2J_{\rm hair}}(2J_{\rm hor}+2J_{\rm hair})^{2k}$. The $(2J_{\rm hor}+2J_{\rm hair})^{2k}$ factor can be expanded in binomial expansion and we get a sum of $2k+1$ different terms. However only the $(2J_{\rm hair})^{2k}$ term will give a non-vanishing contribution to the trace. This is due to the fact that typically the $4k$ fermion zero modes associated with the $4k$ broken supersymmetry generators are all part of the hair degrees of freedom [4]. Saturating each pair of fermon zero modes requires a factor of $(2J_{\rm hair})$; thus we need $2k$ factors of $2J_{\rm hair}$ to saturate the fermion zero modes associated with the hair degrees of freedom. As a result on the right hand side of (1.2) we now need to replace $d_{\rm hair}$ by the helicity trace $B_{2k;hair}$ involving the hair degrees of freedom, and $d_{\rm hor}$ by the Witten index $Tr(-1)^{2J_{\rm hor}}$ over the horizon degrees of freedom.

Now QEF computes the degeneracy of states of a fixed angular momentum, just as it computes the degeneracy of states of a fixed charge, since from the point of view of the two dimensional string theory living on AdS_2 the angular momentum can be regarded as a component of the electric charge vector [36]. Let us denote by $d_{\rm hor}(\vec{q}, J)$ the degeneracy computed using QEF for fixed charges $\vec{q}$ and a fixed angular momentum J along the 3-direction. Then the contribution from angular momentum J to the Witten index associated with the horizon is $(-1)^{2J} d_{\rm hor}(\vec{q}, J)$. Now in four dimensions only $J = 0$ black holes are supersymmetric and contribute to the index. Any other extremal black hole with $J \neq 0$ will break all supersymmetries and hence will not contribute to the index. This gives $d_{\rm hor}(\vec{q}, J = 0)$ as the contribution to the Witten index from the horizon, and multiplying this by B_{2k} associated with the hair degrees of freedom we get the full contribution to the

JHEP08(2009)068

helicity trace index. Thus eq. (1.1), (1.2) can be replaced by

$$B_{2k;\rm micro}(\vec{q}) = B_{2k;\rm macro}(\vec{q})\,, \tag{5.1}$$

$$B_{2k;\rm macro}(\vec{q}) = \sum_n \sum_{\substack{\{\vec{q}_i\},\vec{q}_{\rm hair} \\ \sum_{i=1}^n \vec{q}_i+\vec{q}_{\rm hair}=\vec{q}}} \left\{\prod_{i=1}^n d_{\rm hor}(\vec{q}_i, J_i=0)\right\} B_{2k;\rm hair}(\vec{q}_{\rm hair};\{\vec{q}_i\})\,. \tag{5.2}$$

Since $d_{\rm hor}(\vec{q}_i, J_i = 0)$ is computed by QEF, eqs. (5.1), (5.2) provide a way to compare the helicity trace index in the microscopic description to the QEF in the macroscopic description.

This point of view suggests that while comparing the indices on the microscopic and the macroscopic sides we can not only compare their magnitudes but also their signs. For example the sign of the helicity trace index $-B_6$ for quarter BPS states in a class of $\mathcal{N} = 4$ supersymmetric string theories was calculated in [37] and was found to be positive, at least in the limit when the charges are large. We can compare this with the macroscopic result for the index as follows. For simplicity we shall consider the $\mathcal{N} = 4$ supersymmetric string theory obtained by compactifying type IIB string theory on $K3\times T^2$, and choose a black hole that carries only D-brane charges (D5-branes wrapped on 5-cycles, D3-branes wrapped on 3-cycles and D1-branes wrapped on 1-cycles of $K3 \times T^2$) without any momentum, fundamental string winding, KK monopole or H-monopole charges. In this case we expect that the only hair degrees of freedom are the fermion zero modes associated with the 12 broken supersynmmetry generators, contributing a factor of unity to $-B_{6;hair}$, since classical fluctuations of closed string fields around the black hole background are not expected to produce any D-brane charges. On the other hand since $d_{\rm hor}(\vec{q}, J = 0)$ measures the ground state degeneracy of the dual CFT_1 carrying quantum numbers $(\vec{q}, J = 0)$, it is a positive number. Thus $-B_{6;macro}$ computed from eq. (5.2) is positive, in agreement with the microscopic result.

Another consequence of this viewpoint is that the sum over various configurations appearing on the right hand side of (5.2) involves only those configurations for which the index associated with the hair is non-vanishing. In particular any configuration with accidental fermion zero modes, besides the ones associated with the broken supersymmetry generators, will have vanishing index and hence will not contribute to the sum. Such configurations include for example multi-black hole solutions in $\mathcal{N} = 4$ supersymmetric string theories with at least one center describing a large black hole. These are known not to contribute to the index and hence are expected to have accidental fermion zero modes.

This discussion also raises the question as to whether it is possible to directly compare the microscopic and macroscopic degeneracies for a given angular momentum instead of the indices. While this should be possible in principle, the lack of a non-renormalization theorem implies that we need to carry out the microscopic computation directly in a regime of the parameter space where gravity is strong enough to produce a black hole. This is not possible with the currently available techniques. Typically the microscopic spectrum is computed in the weak coupling regime. By the time we turn on the coupling and bring it to a region where the black hole description is appropriate, the detailed information about the spectrum of BPS states in the microscopic theory may be lost except for an appropriate index that is protected by supersymmetry.

JHEP08(2009)068

6 Quantum entropy function for torsion > 1 dyons

In type IIB string theory compactified on $K3\times T^2$, or equivalently in heterotic string theory on T^6, the charges carried by a generic dyon are labelled by a pair of 28 dimensional vectors (Q,P), each belonging to the 28 dimensional Narain lattice Λ_{28} with signature (6,22) [5]. Physically Q and P denote the electric and magnetic charge vectors in the heterotic description. It was shown in [8] that with the help of S-duality transformations any charge vector can be brought to the form:

$$(Q,P) = (\ell\, Q_0, P_0), \quad \ell\in\mathbb{Z}, \quad Q_0, P_0\in\Lambda_{28}, \quad \gcd\{Q_{0i}P_{0j}-P_{0i}Q_{0j}\} = 1\,, \tag{6.1}$$

where Q_{0i}, P_{0i} are the components of the vectors Q_0, P_0 along some primitive basis vectors of the Narain lattice. The integer ℓ is a discrete duality invariant introduced in [6] which we shall refer to as torsion.

Since we shall be working in the type IIB description, it will be useful to understand the interpretation of the charge vectors directly in type IIB string theory. For this we represent T^2 as a product (topologically but not necessarily metrically) of two circles $S^1\times\widetilde{S}^1$ and denote by x^5 and x^4 the coordinates along S^1 and $\widetilde{S}^1$ respectively, both normalized to have period 2π. Furthermore we shall restrict ourselves to configurations carrying only D-brane charges, i.e. D1/D3/D5-branes wrapped on S^1 or $\widetilde{S}^1$ times 0/2/4 cycles of $K3$. In this case the magnetic charge vector P measures winding numbers of various branes along S^1 and the electrtic charge vector Q measures winding numbers of various branes along $\widetilde{S}^1$. In this limited subspace the charge vectors Q and P are 24 dimensional instead of 28 dimensional, associated with the 24 even cycles of $K3$. There is a natural metric in this 24 dimensional space given by the intersection form of the even cycles of $K3$, and this allows us to define inner products Q^2, P^2 and $Q\cdot P$ among the charge vectors. We shall call these continuous T-duality invariants.

Now one advantage of using pure D-brane configurations is that in this case the hair modes are simple, — they consist of just the twelve fermion zero modes associated with broken supersymmetry generators, and do not carry any charge. This is due to the fact that we do not expect classical fluctuations of closed string fields to carry RR charges. Thus the hair modes give a contribution of 1 to $-B_{6,hair}$ and $d_{\rm hor}(Q,P)$ can be equated to $-B_6$ of the full black hole. This can then be compared with the microscopic result for $-B_6$. The latter was computed in [9–11] and takes the form:

$$(-1)^{Q\cdot P+1}\sum_{s|\ell} s\, f(Q^2/s^2, P^2, Q\cdot P/s)\,, \tag{6.2}$$

where $(-1)^{Q\cdot P+1}f(Q^2,P^2,Q\cdot P)$ denotes the $-B_6$ index of a dyon of charge (Q,P) with $\gcd\{Q_iP_j-Q_jP_i\}=1$. Since the result for the index depends on the domain in which the asymptotic values of the moduli field lie [6, 38–40], (6.2) makes sense only if we specify the domain. We show in appendix A that (6.2) holds if for the s-th term we take the asymptotic moduli to coincide with the attractor values for the charges $(Q/s,P)$. In this case f appearing in (6.2) can be regarded as the index associated with single centered black holes [39]. The explicit form of $f(Q^2,P^2,Q\cdot P)$ involves a Fourier transform of the inverse of the Siegel modular form [15–19], and has been given in appendix A.

JHEP08(2009)068

Our goal in this section will be to get an understanding of (6.2) from the quantum entropy function. The function $f(Q^2, P^2, Q\cdot P)$ has the property that for large Q^2, P^2, $Q\cdot P$ it behaves as $\exp(\pi\sqrt{Q^2P^2-(Q\cdot P)^2})$. Thus the s-th term in the sum will behave as $\exp(\pi\sqrt{Q^2P^2-(Q\cdot P)^2}/s)$. The leading contribution in the large charge limit, coming from the $s=1$ term, matches the exponential of the Wald entropy $\pi\sqrt{Q^2P^2-(Q\cdot P)^2}$, which, as we have seen, can also be regarded as the classical contribution to QEF. This suggests that s-th term in the sum should arise from the contribution to QEF from a new saddle point — satisfying the boundary condition described in (3.14) — whose contribution to the finite part of the exponent in (3.2) is $1/s$ times that of the original $AdS_2\times S^2$ background. Our goal will be find these saddle points. A non-trivial check will be that these new saddle points should exist only for values of s which divide the integer ℓ defined in (6.1).

We begin by writing down the euclidean near horizon metric associated with this black hole. The requirement of $\mathrm{SL}(2,R)$ symmetry, together with the usual rotational and translational symmetries, fixes the ten dimensional metric of type IIB string theory to be of the form

$$ds^2 = v\left(\frac{dr^2}{r^2-1}+(r^2-1)\,d\theta^2\right)+w(d\psi^2+\sin^2\psi d\phi^2)+\frac{R^2}{\tau_2}\left|dx^4+\tau dx^5\right|^2+\widehat{g}_{mn}(\vec{u})du^m du^n\,, \tag{6.3}$$

where $\widehat{g}_{mn}$ and $\vec{u}$ denote the metric and the coordinates along $K3$, v, w, R are real constants and $\tau=\tau_1+i\tau_2$ is a complex constant. (r,θ) label an Euclidean AdS_2 space whereas (ψ,ϕ) label a 2-sphere. Besides these the background contains fluxes of various RR fields. In the six dimensional description, in which all the RR field strengths can be regarded as self-dual or anti-self-dual 3-forms after dimensional reduction on K3, Q represents RR fluxes through the 3-cycle spanned by (x^5,ψ,ϕ) and P represents RR fluxes through the 3-cycle spanned by (x^4,ψ,ϕ). The (anti-)self-duality constraints on the various components of the RR fields in six dimensions relate the fluxes through the (x^4,r,θ) and (x^5,r,θ) planes to those through the (x^5,ψ,ϕ) and (x^4,ψ,ϕ) planes.

Since Q according to (6.1) is ℓ times a primitive vector, the RR fluxes through the cycle spanned by (x^5,ψ,ϕ) is ℓ times a primitive flux. Let us now consider an orbifold of the background (6.3) by the transformation

$$(\theta,\phi,x^5)\to\left(\theta+\frac{2\pi}{s},\phi+\frac{2\pi}{s},x^5+\frac{2\pi k}{s}\right),\quad k,s\in\mathbb{Z},\quad \gcd(s,k)=1\,. \tag{6.4}$$

Since the circle parametrized by x^5 is non-contractible, this is a freely acting orbifold. At the origin $r=1$ of the AdS_2 space we have a non-contractible 3-cycle spanned by (x^5,ψ,ϕ), with the identification $(x^5,\psi,\phi)=(x^5+2\pi k/s,\psi,\phi+2\pi/s)$. As a result of this identification the total flux of RR fields through this cycle is $1/s$ times the original flux. Since the flux quantization constraints require the fluxes through this new 3-cycle to be integers, we see that this orbifold is an allowed configuration in string theory only when ℓ/s is an integer.

We shall now show, following [20], that this configuration has the same asymptotic behaviour as (6.3) up to allowed correction terms of the type described in (3.14) and hence

JHEP08(2009)068

must be included as a new saddle point in the path integral that computes the QEF. For this we take $(s\theta, r/s)$ to be our new (θ, r) coordinates. This generates the metric

$$\begin{aligned} ds^2 &= v\left(\frac{dr^2}{r^2 - s^{-2}} + (r^2 - s^{-2})\, d\theta^2\right) + w(d\psi^2 + \sin^2\psi d\phi^2) \\ &\quad + \frac{R^2}{\tau_2}\left|dx^4 + \tau dx^5\right|^2 + \widehat{g}_{mn}(\vec{u}) du^m du^n\,, \\ (\theta, \phi, x^5) &\equiv \left(\theta + 2\pi, \phi + \frac{2\pi}{s}, x^5 + \frac{2\pi k}{s}\right), \quad s|\ell\,. \end{aligned} \tag{6.5}$$

The RR field strengths in the new coordinate system remain identical to those in the original coordinates. Thus for large r the metric, RR field strengths and the θ periodicity has the same structure as (6.3) but the ϕ and x^5 coordinates are twisted by $2\pi/s$ and $2\pi k/s$ respectively as we go around the θ circle. These can be regarded as the effect of switching on constant Wilson lines at ∞ for the gauge fields associated with ϕ and x^5 translation symmetries [20], *without changing the asymptotic values of the field strengths, i.e. charges.* Since in the path integral we must integrate over the constant modes of the gauge fields keeping the charges fixed, (6.5) represents an allowed configuration in the path integral *for the same values of the charges for which the original solution (6.3) is given.*

Since this new configuration is obtained by taking a $\mathbb{Z}_s$ quotient of the original configuration, the action associated with this configuration will be $1/s$ times the original action. But due to the rescaling of the r coordinate the cutoff r_0 on the new coordinate r will correspond to a cutoff sr_0 on the original radial coordinate. The net result is that the infrared divergent part of the action, being proportional to r_0, is unchanged but the finite part gets scaled by $1/s$. Thus in the large charge limit the leading contribution to the QEF from this saddle point goes as $\exp(\pi\sqrt{Q^2P^2 - (Q\cdot P)^2}/s)$. This observation, together with the fact that these orbifolds exist only when s divides ℓ, makes this saddle point an ideal candidate that contributes to the s-th term in the sum in (6.1).

There is however a subtle issue related to this analysis. The orbifold operation (6.4) breaks the $\mathrm{SL}(2,R)\times\mathrm{SU}(2)$ isometry of $AdS_2\times S^2$ to a $\mathrm{U}(1)\times\mathrm{U}(1)$ subgroup. Thus we actually have a family of such orbifolds parametrized by the points on $(\mathrm{SL}(2,R)/\mathrm{U}(1))\times(\mathrm{SU}(2)/\mathrm{U}(1))$. Since $\mathrm{SL}(2,R)/\mathrm{U}(1)$ has infinite volume, it would suggest that the contribution from this saddle point is multiplied by an infinite factor. However we should note that the orbifold operation also breaks half of the eight supersymmetries of the original background, and hence has four fermion zero modes. Integration over these fermion zero modes produce zero result. This is precisely the problem analyzed in [41] who showed that by suitably regularizing the action one gets a cancellation between the infinities coming from the bosonic zero mode integrals and the zeroes coming from the fermion zero mode integrals. Thus one gets finite contribution from the saddle points described in (6.5).

By examining the symmetries of various terms we can get further evidence for the identification of the s-th term in (6.2) with the contribution from the orbifold (6.4). We first note that the function $f(Q^2, P^2, Q\cdot P)$ is invariant under an $\mathrm{SL}(2,\mathbb{Z})$ transformation

JHEP08(2009)068

on the charges

$$\begin{pmatrix} Q \\ P \end{pmatrix} \to \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} Q \\ P \end{pmatrix}, \quad a, b, c, d \in \mathbb{Z}, \quad ad - bc = 1\,, \tag{6.6}$$

which gives

$$\begin{pmatrix} Q^2 \\ P^2 \\ Q \cdot P \end{pmatrix} \to \begin{pmatrix} a^2 & b^2 & 2\,a\,b \\ c^2 & d^2 & 2\,c\,d \\ a\,c & b\,d & a\,d + b\,c \end{pmatrix} \begin{pmatrix} Q^2 \\ P^2 \\ Q \cdot P \end{pmatrix}. \tag{6.7}$$

Thus the s-th term in (6.2), proportional to $f(Q^2/s^2, P^2, Q \cdot P/s)$, will be invariant under the transformation (6.7) if we restrict a, b, c, d to saisfy

$$a, c, d \in \mathbb{Z}, \quad b \in s\mathbb{Z}, \quad ad - bc = 1\,. \tag{6.8}$$

This describes a $\Gamma^0(s)$ subgroup of $\mathrm{SL}(2, \mathbb{Z})$. We shall now show that this is precisely the symmetry of the contribution from the orbifold (6.4) *if we assume that once a saddle point has been fixed, the contribution depends only on the continuous T-duality invariants* Q^2, P^2 *and* $Q \cdot P$ *and not on the arithmetical properties of the charges.* To this end, note that since Q and P represent winding charges of various branes along the x^4 and x^5 directions respectively, the $\mathrm{SL}(2, \mathbb{Z})$ transformation acts on these coordinates as

$$\begin{pmatrix} x^4 \\ x^5 \end{pmatrix} \to \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x^4 \\ x^5 \end{pmatrix}. \tag{6.9}$$

Thus acting on an orbifold given in (6.4), it takes it to another orbifold with the same shift symmetries on the (θ, ϕ) coordinates, but a new set of shifts on the (x^4, x^5) coordinates:

$$\begin{pmatrix} x^4 \\ x^5 \end{pmatrix} \to \begin{pmatrix} x^4 \\ x^5 \end{pmatrix} + 2\,\pi \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 \\ k/s \end{pmatrix} = \begin{pmatrix} x^4 \\ x^5 \end{pmatrix} + 2\,\pi \begin{pmatrix} bk/s \\ dk/s \end{pmatrix}. \tag{6.10}$$

Now with the restrictions described in (6.8), b/s is an integer and hence the shift of x^4 in (6.10) is trivial. On the other hand we see from (6.8) that d and s must be coprime. Since k and s are also coprime, we see that $d\,k$ and s are coprime. Thus the shifts in (x^4, x^5) given in (6.10) are of the same type as the one appearing in (6.4), with k replaced by a new integer $k' = dk$ coprime to s. Thus the sum of the contributions from all orbifolds of the type given in (6.4), with k running over different values, should be invariant under the $\Gamma^0(s)$ transformation on Q^2, P^2 and $Q \cdot P$ described in (6.7), (6.8). This establishes that both the s-th term in (6.2) and the contribution from the orbifold (6.4) are invariant under the same symmetry group $\Gamma^0(s)$.

Finally we should add a word of caution. While our analysis identifies a specific class of extra contributions to QEF for dyons of torsion > 1, we have not shown that these are the only additional contributions. There may be other effects which also contribute to the difference between the degeneracy formulæ for the dyons of torsion 1 and dyons of torsion> 1. As an example we would like to mention $\mathbb{Z}_s$ orbifolds of the type discussed in (6.4), but with $k = 0$, i.e. without any shift along the x^5 coordinate. These orbifolds

JHEP08(2009)068

have codimension four fixed planes sitting at the origin of AdS_2 and the north or the south pole of S^2. If we now consider the plane spanned by the (x^5, ψ, ϕ) coordinates and sitting at the origin of AdS_2, then the total RR flux through this plane is given by Q/s as discussed below (6.4). However this does not require Q to be quantized in units of s since there may be additional RR flux sitting at the fixed points at $\psi = 0, \pi$ through which the above plane is required to pass. Thus these orbifolds exist for all Q and could be responsible for the subleading contribution to the entropy of quarter BPS dyons even for $\ell = 1$ states [20]. Nevertheless the details of the contribution from this fixed point could depend on whether Q/s is an integer or not since for integer Q/s one does not need to have any RR flux at the fixed points. This then would be an additional source for the extra contributions to the degeneracies of dyons of torsion > 1 compared to the dyons of torsion 1.

Acknowledgments

We would like to thank Nabamita Banerjee, Shamik Banerjee, Borun Chowdhury, Atish Dabholkar, Justin David, Rajesh Gopakumar, Chethan Gowdigere, Rajesh Gupta, Dileep Jatkar, Ipsita Mandal, Samir Mathur, Shiraz Minwalla and Yogesh Srivastava for useful discussions.

A The dyon degeneracy formula for type IIB string theory on $K3 \times T^2$

According to [10] the index $-B_6$ of a torsion ℓ dyon carrying charges of the form $(Q = \ell\, Q_0, P)$ is given by

$$d(Q,P) = (-1)^{Q\cdot P+1} \sum_{s|\ell} s^4 \int_{iM_1-1/2}^{iM_1+1/2} d\check\rho \int_{iM_2-1/(2s^2)}^{iM_2+1/(2s^2)} d\check\sigma \int_{iM_3-1/(2s)}^{iM_3+1/(2s)} d\check v \; e^{-i\pi(\check\sigma Q^2+\check\rho P^2+2\check v Q\cdot P)}\, \Phi_{10}\left(\check\rho, s^2\check\sigma, s\check v\right)^{-1}\,, \tag{A.1}$$

where $\Phi_{10}(\rho,\sigma,v)$ is the Igusa cusp form and

$$M_1 = 2\Lambda \frac{Q^2}{\sqrt{Q^2P^2-(Q\cdot P)^2}}\,, \quad M_2 = 2\Lambda \frac{P^2}{\sqrt{Q^2P^2-(Q\cdot P)^2}}\,, \quad M_3 = -2\Lambda \frac{Q\cdot P}{\sqrt{Q^2P^2-(Q\cdot P)^2}}. \tag{A.2}$$

Here Λ is a sufficiently large positive real number. The choice of integration contour given in (A.2) is valid if we choose the asymptotic moduli to be at the attractor point corresponding to the charges (Q,P). For this choice of the integration contour we pick up the contribution from single centered black holes only [39].

Now making a change of variables

$$\rho = \check\rho, \qquad \sigma = s^2\check\sigma, \qquad v = s\check v\,, \tag{A.3}$$

we can express (A.1) as

$$d(Q,P) = (-1)^{Q\cdot P+1} \sum_{s|\ell} s \int_{iM_1-1/2}^{iM_1+1/2} d\rho \int_{is^2M_2-1/2}^{is^2M_2+1/2} d\sigma \int_{isM_3-1/2}^{isM_3+1/2} dv \; e^{-i\pi(\sigma Q^2/s^2+\rho P^2+2vQ\cdot P/s)}\, \Phi_{10}\left(\rho,\sigma,v\right)^{-1}\,. \tag{A.4}$$

Since for torsion 1 dyons only the $s = 1$ term in (A.4) contributes we see that the s-th term has the structure of the degeneracy of a torsion 1 dyon with continuous T-duality invariants $(Q^2/s^2, P^2, Q\cdot P/s)$. The only issue is whether the choice of integration contour agrees with that relevant for the single centered black holes carrying the invariants $(Q^2/s^2, P^2, Q\cdot P/s)$. To this end we note that the choice of contour for the s-th term in (A.4) may be expressed as

$$M_1 = 2\,\widetilde{\Lambda}\,\frac{(Q/s)^2}{\sqrt{(Q/s)^2P^2 - ((Q/s)\cdot P)^2}}\,, \quad s^2\,M_2 = 2\,\widetilde{\Lambda}\,\frac{P^2}{\sqrt{(Q/s)^2P^2 - ((Q/s)\cdot P)^2}}\,,$$
$$s\,M_3 = -2\,\widetilde{\Lambda}\,\frac{(Q/s)\cdot P}{\sqrt{(Q/s)^2P^2 - ((Q/s)\cdot P)^2}}, \qquad \widetilde{\Lambda} \equiv s\,\Lambda\,. \tag{A.5}$$

Comparing this with (A.2) we see that this is indeed the correct choice of contour for picking up contribution from single centered black holes carrying invariants $(Q^2/s^2, P^2, Q\cdot P/s)$. Thus we can express (A.4) as

$$(-1)^{Q\cdot P+1}\sum_{s|\ell} f(Q^2/s^2, P^2, Q\cdot P/s)\,, \tag{A.6}$$

where $(-1)^{Q\cdot P+1} f(Q^2, P^2, Q\cdot P)$ represents the index of a single centered black hole of torsion 1 and continuous T-duality invariants $(Q^2, P^2, Q\cdot P)$.

References

[1] A. Sen, *Entropy Function and AdS_2/CFT_1 Correspondence*, *JHEP* **11** (2008) 075 [arXiv:0805.0095] [SPIRES].

[2] A. Sen, *Quantum Entropy Function from AdS_2/CFT_1 Correspondence*, arXiv:0809.3304 [SPIRES].

[3] A. Strominger and C. Vafa, *Microscopic Origin of the Bekenstein-Hawking Entropy*, *Phys. Lett.* **B 379** (1996) 99 [hep-th/9601029] [SPIRES].

[4] N. Banerjee, I. Mandal and A. Sen, *Black Hole Hair Removal*, *JHEP* **07** (2009) 091 [arXiv:0901.0359] [SPIRES].

[5] K.S. Narain, *New Heterotic String Theories in Uncompactified Dimensions < 10*, *Phys. Lett.* **B 169** (1986) 41 [SPIRES].

[6] A. Dabholkar, D. Gaiotto and S. Nampuri, *Comments on the spectrum of CHL dyons*, *JHEP* **01** (2008) 023 [hep-th/0702150] [SPIRES].

[7] S. Banerjee and A. Sen, *Duality Orbits, Dyon Spectrum and Gauge Theory Limit of Heterotic String Theory on T^6*, *JHEP* **03** (2008) 022 [arXiv:0712.0043] [SPIRES].

[8] S. Banerjee and A. Sen, *S-duality Action on Discrete T-duality Invariants*, *JHEP* **04** (2008) 012 [arXiv:0801.0149] [SPIRES].

[9] S. Banerjee, A. Sen and Y.K. Srivastava, *Generalities of Quarter BPS Dyon Partition Function and Dyons of Torsion Two*, *JHEP* **05** (2008) 101 [arXiv:0802.0544] [SPIRES].

[10] S. Banerjee, A. Sen and Y.K. Srivastava, *Partition Functions of Torsion > 1 Dyons in Heterotic String Theory on T^6*, *JHEP* **05** (2008) 098 [arXiv:0802.1556] [SPIRES].

JHEP08(2009)068

[11] A. Dabholkar, J. Gomes and S. Murthy, *Counting all dyons in $N = 4$ string theory*, arXiv:0803.2692 [SPIRES].

[12] G.W. Moore, *Attractors and arithmetic*, hep-th/9807056 [SPIRES].

[13] G.W. Moore, *Arithmetic and attractors*, hep-th/9807087 [SPIRES].

[14] G.W. Moore, *Les Houches lectures on strings and arithmetic*, hep-th/0401049 [SPIRES].

[15] R. Dijkgraaf, E.P. Verlinde and H.L. Verlinde, *Counting Dyons in $N = 4$ String Theory*, *Nucl. Phys.* **B 484** (1997) 543 [hep-th/9607026] [SPIRES].

[16] G. Lopes Cardoso, B. de Wit, J. Kappeli and T. Mohaupt, *Asymptotic degeneracy of dyonic $N = 4$ string states and black hole entropy*, *JHEP* **12** (2004) 075 [hep-th/0412287] [SPIRES].

[17] D. Shih, A. Strominger and X. Yin, *Recounting dyons in $N = 4$ string theory*, *JHEP* **10** (2006) 087 [hep-th/0505094] [SPIRES].

[18] D. Gaiotto, *Re-recounting dyons in $N = 4$ string theory*, hep-th/0506249 [SPIRES].

[19] J.R. David and A. Sen, *CHL dyons and statistical entropy function from $D1 - D5$ system*, *JHEP* **11** (2006) 072 [hep-th/0605210] [SPIRES].

[20] N. Banerjee, D.P. Jatkar and A. Sen, *Asymptotic Expansion of the $N = 4$ Dyon Degeneracy*, *JHEP* **05** (2009) 121 [arXiv:0810.3472] [SPIRES].

[21] S.S. Gubser, I.R. Klebanov and A.M. Polyakov, *Gauge theory correlators from non-critical string theory*, *Phys. Lett.* **B 428** (1998) 105 [hep-th/9802109] [SPIRES].

[22] E. Witten, *Anti-de Sitter space and holography*, *Adv. Theor. Math. Phys.* **2** (1998) 253 [hep-th/9802150] [SPIRES].

[23] J.M. Maldacena, *The large-N limit of superconformal field theories and supergravity*, *Adv. Theor. Math. Phys.* **2** (1998) 231 [*Int. J. Theor. Phys.* **38** (1999) 1113] [hep-th/9711200] [SPIRES].

[24] A. Strominger, *AdS_2 quantum gravity and string theory*, *JHEP* **01** (1999) 007 [hep-th/9809027] [SPIRES].

[25] A. Gregori et al., *R^2 corrections and non-perturbative dualities of $N = 4$ string ground states*, *Nucl. Phys.***B 510** (1998) 423 [hep-th/9708062] [SPIRES].

[26] E. Kiritsis, *Introduction to non-perturbative string theory*, hep-th/9708130 [SPIRES].

[27] R.M. Wald, *Black hole entropy is the Noether charge*, *Phys. Rev.* **D 48** (1993) 3427 [gr-qc/9307038] [SPIRES].

[28] T. Jacobson, G. Kang and R.C. Myers, *On black hole entropy*, *Phys. Rev.* **D 49** (1994) 6587 [gr-qc/9312023] [SPIRES].

[29] V. Iyer and R.M. Wald, *Some properties of Noether charge and a proposal for dynamical black hole entropy*, *Phys. Rev.* **D 50** (1994) 846 [gr-qc/9403028] [SPIRES].

[30] T. Jacobson, G. Kang and R.C. Myers, *Black hole entropy in higher curvature gravity*, gr-qc/9502009 [SPIRES].

[31] A. Sen, *Black Hole Entropy Function and the Attractor Mechanism in Higher Derivative Gravity*, *JHEP* **09** (2005) 038 [hep-th/0506177] [SPIRES].

[32] P. Kraus and F. Larsen, *Microscopic Black Hole Entropy in Theories with Higher Derivatives*, *JHEP* **09** (2005) 034 [hep-th/0506176] [SPIRES].

JHEP08(2009)068

[33] A. Castro, D. Grumiller, F. Larsen and R. McNees, *Holographic Description of AdS_2 Black Holes*, *JHEP* **11** (2008) 052 [arXiv:0809.4264] [SPIRES].

[34] T. Hartman and A. Strominger, *Central Charge for AdS_2 Quantum Gravity*, *JHEP* **04** (2009) 026 [arXiv:0803.3621] [SPIRES].

[35] M. Alishahiha and F. Ardalan, *Central Charge for 2D Gravity on AdS_2 and AdS_2/CFT_1 Correspondence*, *JHEP* **08** (2008) 079 [arXiv:0805.1861] [SPIRES].

[36] D. Astefanesei, K. Goldstein, R.P. Jena, A. Sen and S.P. Trivedi, *Rotating attractors*, *JHEP* **10** (2006) 058 [hep-th/0606244] [SPIRES].

[37] A. Sen, *Black Hole Entropy Function, Attractors and Precision Counting of Microstates*, *Gen. Rel. Grav.* **40** (2008) 2249 [arXiv:0708.1270] [SPIRES].

[38] A. Sen, *Walls of Marginal Stability and Dyon Spectrum in $N = 4$ Supersymmetric String Theories*, *JHEP* **05** (2007) 039 [hep-th/0702141] [SPIRES].

[39] M.C.N. Cheng and E. Verlinde, *Dying Dyons Don't Count*, *JHEP* **09** (2007) 070 [arXiv:0706.2363] [SPIRES].

[40] M.C.N. Cheng and E.P. Verlinde, *Wall Crossing, Discrete Attractor Flow and Borcherds Algebra*, arXiv:0806.2337 [SPIRES].

[41] C. Beasley et al., *Why $Z_{BH} = |Z_{top}|^2$*, hep-th/0608021 [SPIRES].

JHEP08(2009)068